

Renormalization Group Analysis of October Market Crashes

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The self-similar analysis of time series, suggested earlier by the authors, is applied to the description of market crises. The main attention is payed to the *October 1929, 1987 and 1997* stock market crises, which can be successfully treated by the suggested approach. The analogy between market crashes and critical phenomena is emphasized.

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I. SELF-SIMILAR ANALYSIS

Renormalization group approach is known to be a powerful tool for treating critical phenomena in statistical physics. An interesting example of complex statistical systems are markets [1-5], and market crashes are somewhat analogous to critical phenomena [1,6]. Keeping in mind this analogy, we have recently proposed [7] that the time series describing stock-market crises can be treated by means of resummation or renormalization methods of theoretical physics. The method we suggested [7] is based on the algebraic self-similar renormalization [8-10] which is a specific variant of the self-similar approximation theory [11-15]. The self-similar analysis developed in our previous paper [7] for treating stock-market crises was shown to describe well a number of such crises from the past. Also, we attempted to predict the behaviour of some stock-market indices at the end of *October 1997*, before the so-called market correction occurred. Thus, for the NYSE Composite index we predicted the value 478.855. Now we know that the actual value of this index on *October 31* was 481.14. So, the error of our forecast is only -0.47% . For the Standard and Poor 500 index we found the value 935.082, while its actual value on *October 31* was 914.62. Consequently, the error is 2.24% . And for the Dow Jones index we predicted the value 7788, which, as compared to the realized value 7442.07, makes the error of 4.65% . This shows that we correctly predicted the fall of these stock-market indices before it actually occurred at the end of *October 1997*.

In the present paper we use the self-similar analysis [7] for considering in detail the famous *October 1929* and *October 1987* stock market crashes as well as the *October 1997* crisis. The general scheme of the method has been thoroughly described in Ref. [7], because of which we shall not repeat it here, in full, but, for convenience, we will remind the main steps necessary for the analysis.

Assume that we are considering a function of time, $f(t)$, which characterizes the market activity. For instance, $f(t)$ can be the price of some security or commodity, or it can be some price index. Let the values of $f(t)$ be known for n equidistant successive moments of time, $t = 0, 1, 2, \dots, n-1$, preceding a crash,

$$f(0) = a_0, \quad f(1) = a_1, \quad \dots, \quad f(n-1) = a_{n-1}. \quad (1)$$

Our aim is to find $f(n)$ at the time $t = n$. The set of data (1) can be presented in the form of the polynomial

$$f(t) = \sum_{k=0}^{n-1} A_k t^k \quad (0 \leq t \leq n-1), \quad (2)$$

with the coefficients A_k to be determined from the set of equations (1). The polynomial representation (2) means that there are the following n approximations for the function $f(t)$ considered:

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$$p_0(t) = A_0 = a_0, \quad p_1(t) = p_0(t) + A_1 t, \quad \dots, \quad p_{n-1}(t) = p_{n-2}(t) + A_{n-1} t^{n-1}. \quad (3)$$

For the sequence $\{p_k(t)\}$, we construct the dynamical system called the approximation cascade, $\{y_k\}$, whose trajectory $\{y_k(\varphi, s)\}$ consists of the points $y_k(\varphi, s) \equiv P_k(t(\varphi, s), s)$, such that $P_k(t, s) \equiv t^s p_k(t)$ and the function $t(\varphi, s) = (\varphi/a_0)^{1/s}$ is defined by the equation $P_0(t, s) = a_0 t^s = \varphi$. The evolution equation for the approximation cascade $\{y_k\}$ can be written in the functional form, $y_{k+p}(\varphi, s) = y_k(y_p(\varphi, s), s)$, or in the integral form yielding

$$\int_{P_{k-1}}^{P_k^*} \frac{d\varphi}{v_k(\varphi, s)} = \tau, \quad v_k(\varphi, s) = y_k(\varphi, s) - y_{k-1}(\varphi, s),$$

where $P_k = P_k(t, s)$ and $P_k^* = P_k^*(t, s, \tau)$ is a quasifixed point, with τ being the minimal time necessary for reaching the quasifixed point P_k^* . Finding the latter and accomplishing the transform $\lim_{s \rightarrow \infty} t^{-s} P_k^*(t, s, \tau)$ for each $k = 1, 2, \dots, n-1$, as is prescribed by the self-similar bootstrap [10], we come to the sequence of the self-similar exponential approximants

$$f_k^*(t, \tau) = A_0 \exp \left(\frac{A_1}{A_0} t \exp \left(\frac{A_2}{A_1} t \dots \exp \left(\frac{A_k}{A_{k-1}} \tau t \right) \right) \dots \right). \quad (4)$$

The local stability, or the local convergence [15], of the sequence $\{f_k^*(t, \tau)\}$ is characterized by the local multipliers

$$M_k(t, \tau) \equiv \frac{\delta f_k^*(t, \tau)}{\delta f_1^*(t, 1)}. \quad (5)$$

The practical way of calculating the latter is as follows: From the equation $f_1^*(t, 1) = \varphi$, we define the function $t(\varphi) = (A_0/A_1) \ln(\varphi/A_0)$; then, introducing $z_k(\varphi, \tau) \equiv f_k^*(t(\varphi), \tau)$, we may write

$$M_k(t, \tau) = \left[\frac{\partial z_k(\varphi, \tau)}{\partial \varphi} \right]_{\varphi=f_1^*(t, 1)}.$$

An important particular case is when $\tau = 1$, giving the multiplier

$$M_k(t) \equiv M_k(t, 1). \quad (6)$$

If $|M_k(n)| < 1$, then the sequence $\{f_k^*(t, 1)\}$ is locally stable at $t = n$, and $f_{n-1}^*(n, 1)$ is to be a reasonable approximation for the function $f(t)$ at $t = n$. When the terms $f_{n-1}^*(n, 1)$ and $f_{n-2}^*(n, 1)$ are noticeably different from each other, this means that we are yet far from the fixed point. One possibility then could be to define a Cesaro average of the corresponding approximations [14,15]. Another option is to locate the quasifixed point by imposing the minimal-difference condition

$$|f_{n-1}^*(t, \tau_n) - f_{n-2}^*(t, \tau_n)| = \min_{\tau} |f_{n-1}^*(t, \tau) - f_{n-2}^*(t, \tau)|, \quad (7)$$

which defines the corresponding effective time $\tau_n = \tau_n(t)$. The simplest variant of condition (7) is the equality $f_{n-1}^*(t, \tau) = f_{n-2}^*(t, \tau)$ resulting in the equation

$$\tau = \exp \left(\frac{A_{n-1}}{A_{n-2}} t \tau \right). \quad (8)$$

Defining $\tau = \tau_n(t)$ from (8) and substituting it into $f_{n-1}^*(t, \tau)$, we obtain a forecast for the time $t \geq n$,

$$f_{n-1}^*(t) \equiv f_{n-1}^*(t, \tau_n(t)). \quad (9)$$

The stability of the fixed point (9) is characterized by the multiplier

$$M_{n-1}^*(t) \equiv \frac{1}{2} [M_{n-1}(t, \tau_n(t)) + M_{n-2}(t, \tau_n(t))]. \quad (10)$$

Another way of defining a quasifixed point is through the average

$$\bar{f}_{n-1}(t) \equiv \frac{1}{2} [f_{n-1}^*(t, 1) + f_{n-2}^*(t, 1)]. \quad (11)$$

This definition is valid even when Eq. (8) has no solution. The multiplier, characterizing the stability of the quasifixed point (11), can be defined as

$$\bar{M}_{n-1}(t) \equiv \frac{1}{2} [M_{n-1}(t, 1) + M_{n-2}(t, 1)]. \quad (12)$$

Varying the number $n = 3, 4, 5, \dots$, we obtain a set of possible forecasts (9) and (11). The *optimal forecast* is, by definition, that corresponding to the minimal absolute value of a multiplier from the family of all available multipliers (10) and (12).

It is worth emphasizing that our main idea of treating market dynamics in the vicinity of a crisis as a self-similar evolution is based on the analogy between market crises and critical phenomena. The collective crowd behaviour of many interacting agents becomes prevailing near a market crisis [6,16]. In the precrisis region, the market dynamics can be described as a superposition of two types of temporal modes. One is the dominant slow mode corresponding to the collective behaviour, and all others are fast modes caused by random individual interactions and external sources. In other words, the dominant collective mode describes the coherent behaviour of strongly correlated market agents, while the subordinated fast modes correspond to the stochastic incoherent motion of these agents. The development of such a coherent behaviour is a necessary condition for the formation of a law of collective motion, which, in turn, can be expressed as a self-similar evolution. It may happen that among the subordinated fast modes there is a hierarchy, so that the slowest among these fast modes, being influenced by the collective behaviour, displays, in the precrisis region, specific features. This, for instance, can have to do with the appearance of the log-periodic oscillations [1,6,17] near financial crashes and near the crisis phenomena in several other systems [18]. Such precursor phenomena are also similar to heterophase fluctuations occurring in statistical systems near phase transitions [19].

In this way, although the property of a market at each time moment is, in general, related to all its previous history [20], but in the vicinity of a crisis, there appears a principally new feature – *collective coherent behaviour*. It is just this collective behaviour that makes it possible to formulate, by means of the self-similar analysis, a law of motion for a market and to forecast crises. And also, it is because of this coherent behaviour, an accurate description of a crash may be achieved with the data for only a few temporal points preceding the crash.

II. OCTOBER CRASHES

Now we pass to the application of the method to the series of the New York Stock Exchange (NYSE) Composite index in the course of the *October 1987* and *October 1997* crisis, when the index changed sharply during the time comparable to the resolution of the time series. The choice of the NYSE Composite index is caused by the easy availability of the data stored in the NYSE Historical Statistical Archive in Internet. For the *October 1929* crash the NYSE Composite index is not available, because of which we consider the time series for the Standard Statistic index.

A. October 1997 Crisis

Considering the corresponding events, we make the self-similar analysis for different number of points. Below, we describe this analysis for the time series of the NYSE Composite index with one month resolution, aiming to forecast the value of the index for *October 31, 1997*.

Three-point analysis. The following historical data are available:

$$a_0 = 494.50 \text{ (July 31, 97)}, \quad a_1 = 470.48, \quad a_2 = 497.23 \text{ (Sept. 30, 97)}.$$

From condition (1), the coefficients of polynomial (2) are $A_0 = a_0$, $A_1 = -49,405$, and $A_2 = 25.385$. For the exponential approximant (4) at $t = 3$ and $\tau = 1$, we have $f_1^*(3, 1) = 366.435$ and $f_2^*(3, 1) = 463.768$. The corresponding multipliers defined in (6) are $M_1(3) = 1$ and $M_2(3) = -0.147$. The minimal-difference condition (7), with $n = 3$ and $t = 3$, gives the effective time $\tau = \tau_3(3) = 0.4784$. Then, the self-similar exponential approximant (9) becomes $f_2^*(3) = 428.447$, and the multiplier (10) is $M_2^*(3) = 0.332$. The averages (11) and (12) are $\bar{f}_2(3) = 415.102$ and $\bar{M}_2(3) = 0.427$, respectively.

Four-point analysis. The dynamics of the considered index from June 30, 1997 to September 30, 1997 is given by the data

$$a_0 = 462.44 \text{ (June 30, 97)}, \quad a_1 = 494.50, \quad a_2 = 470.48, \quad a_3 = 497.23 \text{ (Sept. 30, 97)}.$$

The polynomial coefficients of polynomial (2) are $A_1 = 95.717$, $A_2 = -81.465$, and $A_3 = 17.808$. For approximants (4) we get $f_2^*(4,1) = 475.338$ and $f_3^*(4,1) = 564.892$, with the related multipliers $M_2(4) = 0.106$ and $M_3(4) = -0.036$. From the minimal-difference condition (8), at $t = n = 4$, we find $\tau = \tau_4(4) = 0.5946$. Thus, the self-similar exponential approximant (9) yields $f_3^*(4) = 515.886$, while multiplier (10) gives $M_3^*(4) = 0.032$. For the averages (11) and (12) we have $\bar{f}_3(4) = 520.115$ and $\bar{M}_3(4) = 0.035$.

Five-point analysis. The corresponding data are

$$a_0 = 441.78 \text{ (May 30, 97)}, \quad a_1 = 462.44, \quad a_2 = 494.50, \\ a_3 = 470.48, \quad a_4 = 497.23 \text{ (Sept. 30, 97)}.$$

From condition (1), we find the coefficients of polynomial (2), $A_1 = -51.116$, $A_2 = 119.341$, $A_3 = -54.829$, and $A_4 = 7.264$. Formula (4) gives $f_3^*(5,1) = 369.416$ and $f_4^*(5,1) = 434.65$. The mapping multipliers (6) are $M_3(5) = 1.163$ and $M_4(5) = -0.056$. The minimal-difference condition (8) results in $\tau = 0.6501$. From (9) we have $f_4^*(5) = 423.595$, multiplier (10) being $M_4^*(5) = 0.18$. The averages (11) and (12) are $\bar{f}_4(5) = 402.033$ and $\bar{M}_4(5) = 0.554$.

Six-point analysis. In the same way, from the data

$$a_0 = 416.94 \text{ (Apr. 30, 97)}, \quad a_1 = 441.78, \quad a_2 = 462.44, \\ a_3 = 494.50, \quad a_4 = 470.48, \quad a_5 = 497.23 \text{ (Sept. 30, 97)},$$

we find the polynomial coefficients $A_1 = 104.366$, $A_2 = -155.195$, $A_3 = 98.434$, $A_4 = -24.91$, and $A_5 = 2.145$. Following the standard prescription, we get $f_4^*(6,1) = 430.124$ and $f_5^*(6,1) = 520.109$, with the corresponding multipliers $M_4(6) = -0.022$ and $M_5(6) = 0.03$. The minimal-difference condition (8) gives $\tau = 0.6974$. Thence, for approximant (9) and multiplier (10), we obtain $f_5^*(6) = 478.855$ and $M_5^*(6) = 0.023$, while for (11) and (12), we find $\bar{f}_5(6) = 475.117$ and $\bar{M}_5(6) = 0.004$, respectively.

As is explained in the first section of the paper, the optimal forecast is that corresponding to the minimal modulus of the related multiplier, which means that the found fixed point is the most stable one. In the above case, the optimal forecast is that given by the six-point analysis, $\bar{f}_5(6) = 475.117$, which, compared to the *October 31, 1997* index 484.14, has the error -1.25% .

B. October 1987 Crash

Now we consider the behaviour of the NYSE Composite index before the October 1987 crash. The data are taken with the three month resolution. Our aim is to make a forecast for *October 30, 1987*.

Three-point analysis. The data for the considered index from the first quarter to the third quarter of 1987 are

$$a_0 = 156.11 \text{ (Jan. 30, 87)}, \quad a_1 = 162.86, \quad a_2 = 178.64 \text{ (July 31, 87)}.$$

For the polynomial coefficients, we get $A_1 = 2.235$ and $A_2 = 4.515$. The sequence of exponential approximants is locally unstable, since $M_1(3) = 1$ and $M_2(3) \sim 10^{11}$. As an estimate, the value $f_1^*(3,1) = 162.961$ can be taken.

Four-point analysis. From the fourth quarter of 1986 to the third quarter of 1987, the data are

$$a_0 = 140.42 \text{ (Oct. 31, 86)}, \quad a_1 = 156.11, \quad a_2 = 162.86, \quad a_3 = 178.64 \text{ (July 31, 87)}.$$

For the polynomial coefficients, we have $A_1 = 26.15$, $A_2 = -13.455$, and $A_3 = 2.995$. Repeating the standard steps, we find $f_2^*(4,1) = 154.433$ and $f_3^*(4,1) = 193.381$. The corresponding multipliers are $M_2(4) = -0.071$ and $M_3(4) = 0.255$. From the minimal-difference condition, we get $\tau = 0.591$. Finally, the self-similar exponential approximant (9) is $f_3^*(4) = 175.109$, with the multiplier $M_3^*(4) = 0.018$, while the average (11) becomes $\bar{f}_3(4) = 173.907$, with the multiplier $\bar{M}_3(4) = 0.092$.

Five-point analysis. From the data

$$a_0 = 135.89 \text{ (July 31, 86)}, \quad a_1 = 140.42, \quad a_2 = 156.11,$$

$$a_3 = 162.86, \quad a_4 = 178.64 \text{ (July 31, 87),}$$

we get the polynomial coefficients $A_1 = -17.268$, $A_2 = 33.079$, $A_3 = -12.868$, and $A_4 = 1.586$. Then we find $f_3^*(5, 1) = 115.623$ and $f_4^*(5, 1) = 132.899$, with $M_3(5) = 0.937$ and $M_4(5) = -0.065$. The minimal-difference condition gives $\tau = 0.664$. The resulting self-similar exponential approximant (9) is $f_4^*(5) = 129.821$, with the multiplier $M_4^*(5) = 0.139$, and the average (11) is $\bar{f}_4(5) = 124.261$, with the multiplier $\bar{M}_4(5) = 0.436$.

Six-point analysis. Being based on the data

$$a_0 = 135.75 \text{ (Apr. 30, 86)}, \quad a_1 = 135.89, \quad a_2 = 140.42,$$

$$a_3 = 156.11, \quad a_4 = 162.86, \quad a_5 = 178.64 \text{ (July 31, 86),}$$

we have the polynomial coefficients $A_1 = 19.907$, $A_2 = -40.564$, $A_3 = 26.787$, $A_4 = -6.531$, and $A_5 = 0.541$. In the usual way, we get $f_4^*(6, 1) = 136.656$ and $f_5^*(6, 1) = 147.008$, the related multipliers being $M_4(6) = -0.019$ and $M_5(6) = 0.031$. The minimal-difference condition yields $\tau = 0.7045$. The self-similar exponential approximation (9) is $f_5^*(6) = 141.991$, with the multiplier $M_5^*(6) = 0.03$, while for (11) we get $\bar{f}_5(6) = 141.832$, with the multiplier $\bar{M}_5(6) = 0.006$.

Among all multipliers from the sets $\{M_{n-1}^*(n)\}$ and $\{\bar{M}_{n-1}(n)\}$, with $n = 3, 4, 5, 6$, the multiplier $\bar{M}_5(6)$ has the minimal absolute value. Therefore, as the optimal forecast, we accept $\bar{f}_5(6) = 141.832$. The actual value of the index on *October 30, 1987* was 140.8, so that our forecast differs from it only by 0.73%.

C. October 1929 Crash

The historical data from the League of Nations Statistical Yearbook for the Standard Statistics index of the New York stock market from April 1929 to September 1929, with one month resolution are

$$193 \text{ (Apr.)}, \quad 193 \text{ (May)}, \quad 191 \text{ (June)}, \quad 203 \text{ (July)}, \quad 210 \text{ (Aug.)}, \quad 216 \text{ (Sept.)},$$

where the value for 1926 is taken for 100.

Similarly to the cases expounded above, we find in the three-point analysis $f_2^*(3) = 222.921$ and $M_2^*(3) = 0.761$, in the four-point analysis, we get $f_3^*(4) = 222.916$ and $M_3^*(4) = 0.152$, and in the five-point analysis, we have $f_4^*(5) = 179.503$ and $M_4^*(5) = 0.175$. We present the six-point analysis in more details. In the latter case, the polynomial coefficients are $A_1 = 26.683$, $A_2 = -49.208$, $A_3 = 28.333$, $A_4 = -6.292$, and $A_5 = 0.483$. For the exponential approximants (4), we find $f_4^*(6, 1) = 194.884$ and $f_5^*(6, 1) = 206.722$, with the corresponding multipliers $M_4(6) = -0.025$ and $M_5(6) = 0.021$. From the minimal difference condition, we get $\tau = 0.7182$, so that the exponential approximant (9) becomes $f_5^*(6) = 201.692$, with the related multiplier $M_5^*(6) = 0.021$. For the average approximant (11), we obtain $\bar{f}_5(6) = 200.803$, with the multiplier $\bar{M}_5(6) = -0.004$.

The optimal self-similar exponential approximant is $\bar{f}_5(6) = 200.803$. The actual value of the index in *October 1929* was 194, so that our forecast deviates from this by 3.51%.

III. DISCUSSION

We have shown that by means of the algebraic self-similar renormalization [8-10] it is possible to analyse stock market time series and even to predict market crashes. We have applied the approach to the time series corresponding to the *October 1997* crisis and to the *October 1987* and *October 1929* stock market crashes. All these events, within the self-similar renormalization procedure, are quite similar to each other.

For the 1987 and 1997 crises we have chosen for demonstration the dynamics of the NYSE Composite index. It is worth emphasizing that this choice is not principle, but is just a matter of convenience. The same analysis can be done for other representative indices. To show this, we present here such an analysis for the Nasdaq Composite index during the *October 1997* crisis.

The dynamics of the latter index from April 30, 1997 to September 30, 1997 has been as follows:

$$1260 \text{ (Apr. 30)}, \quad 1390 \text{ (May 31)}, \quad 1440 \text{ (June 30)},$$

Following the same way as above, we have in the three-point analysis $f_1^*(3, 1) = 1486$, $f_2^*(3, 1) = 1591$ and $M_1(3) = 1$, $M_2(3) = -0.086$. Defining from (8) the effective time, we get for the approximant (9) the value $f_2^*(3) = 1559$, with $|M_2^*(3)| = 0.147$. For the averages (11) and (12), we find $\bar{f}_2(3) = 1538.5$ and $\bar{M}_2(3) = 0.457$. Analogously, in the four-point analysis, we have $f_2^*(4, 1) = 1545$, $f_3^*(4, 1) = 1911$ and $M_2(4) = -0.056$, $M_3(4) = 0.148$. The self-similar approximant (9) becomes $f_3^*(4) = 1736$, with $|M_3^*(4)| = 0.016$. And for (11) and (12), we get $\bar{f}_3(4) = 1728$, with $\bar{M}_3(4) = 0.046$. The five-point analysis yields $f_3^*(5, 1) = 1120$, $f_4^*(5, 1) = 1348$ and $M_3(5) = 1.128$, $M_4(5) = -0.072$. Approximant (9) is $f_4^*(5) = 1306$, with $|M_4^*(5)| = 0.197$. Also, $\bar{f}_4(5) = 1234$, with $\bar{M}_4(5) = 0.528$. Finally, the six-point analysis gives $f_4^*(6, 1) = 1358$, $f_5^*(6, 1) = 1820$, with $M_4(6) = -0.01$, $M_5(6) = 0.011$. Then, $f_5^*(6) = 1623$, with $|M_5^*(6)| = 0.008$ and $\bar{f}_5(6) = 1589$, with $\bar{M}_5(6) = 0.0005$.

Comparing all multipliers, we see that $\bar{M}_5(6)$ has the minimal absolute value. Therefore, the optimal forecast is $\bar{f}_5(6) = 1589$. The actual value of the Nasdaq Composite index on October 31, 1997 was 1593.61. Our forecast deviates from this by only -0.29% .

Using for the given data (1) the polynomial representation (2), we obtain a set $\{A_k\}$ of polynomial coefficients. The latter define the tendencies existing in the market, so that positive or negative coefficients correspond to growth or decline, respectively. These tendencies compete with each other, analogously to heterophase fluctuations in statistical systems [19]. The state of a market at each time moment is presented as a superexponential function incorporating the mixture of different tendencies. The optimal state is selected as the most stable one.

The possibility of treating market dynamics near a crisis as a self-similar evolution is based on the analogy between market crisis and critical phenomena. The terms of a time series before a crisis contain a hidden information about this approaching crisis. The self-similar analysis plays the role of a decoder deciphering the hidden information.

The intensity of a crisis is determined by an interplay between the polynomial coefficients, whose positive or negative signs represent two competing tendencies, of growth or decay, respectively. The competition between these two different tendencies makes the market heterogeneous. The heterogeneity of a market is a necessary condition for the occurrence of a crisis, which happens when one of two competing tendencies becomes dominant. This is in complete analogy with phase transitions in heterophase statistical systems [19].

It is worth noting that the suggested approach can be applied only to those markets whose evolution is governed by the collective behaviour of interacting agents. Such markets can be called self-regulated or self-organized. It is only these markets are analogous to complex statistical systems whose behaviour is caused by internal reasons. In the case of some strong external forces acting on a market, it cannot be considered as self-regulated and, consequently, it loses the possibility of being described by self-similar dynamics. If the external influence is not too strong, a market can be rather stochastic at a short-time scale but on average self-similar at a longer time scale [7]. We plan to give a more detailed consideration of these questions in forthcoming publications.

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